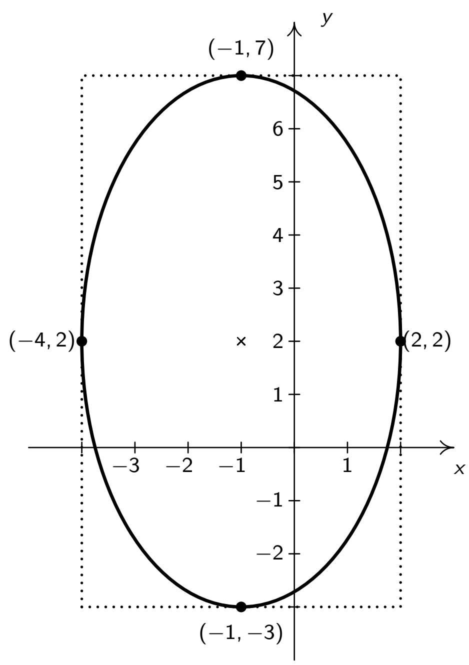
ELLIPSES

EXAMPLE:

1. (a) We rewrite $25(x+1)^2+9(y-2)^2=225$ as follows:

$$25(x+1)^2+9(y-2)^2=225 \to \frac{(x-(-1))^2}{9}+\frac{(y-2)^2}{25}=1 \leftrightarrow \frac{(x-(-1))^2}{(3)^2}+\frac{(y-2)^2}{(5)^2}=1.$$

We identify $h=-1$ and $k=2$, so the center of the ellipse is $(-1,2)$. We have $a=3$ so we move 3 units left and right from the center to obtain two points on the ellipse: $(-4,2)$ and $(2,2)$. Likewise, since $b=5$, we move up and down 5 units from the center to find two more points on the ellipse: $(-1,7)$ and $(-1,-3)$. As an aid to sketching, we draw a rectangle matching this description, called a **guide rectangle**, and sketch the ellipse inside this rectangle as seen below.



The graph of $25(x+1)^2+9(y-2)^2=225$.

Since we moved farther from the center in the y direction than in the x direction, the major axis will lie along the vertical line $x=-1$, while the minor axis lies along the horizontal line $y=2$. The vertices are the points on the ellipse which lie along the major axis so in this case, they are the points $(-1,7)$ and $(-1,-3)$, and the endpoints of the minor axis are $(-4,2)$ and $(2,2)$. (Notice these points are the four points we used to draw the guide rectangle.)

We find $c=\sqrt{25-9}=\sqrt{16}=4$, which means the foci lie 4 units from the center. Since the major axis is vertical, the foci lie 4 units above and below the center, at $(-1,-2)$ and $(-1,6)$.

(b) In the equation $x^2+4y^2-2x+24y+33=0$ we have a sum of two squares with unequal coefficients, it's a good bet we have an ellipse on our hands.

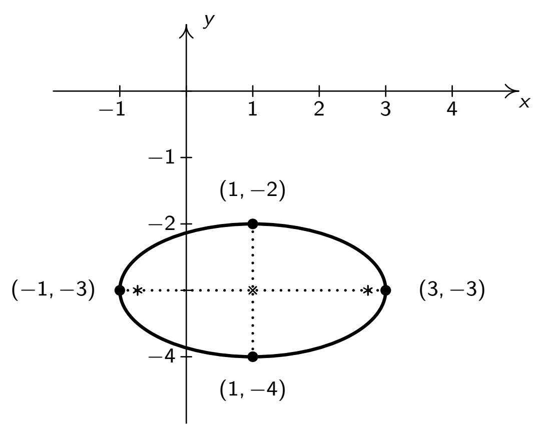
We need to put the equation into standard form to graph:

$$\begin{aligned}
x^2+4y^2-2x+24y+33 &= 0 \\
x^2-2x+4y^2+24y &= -33 && \text{Subtract 33 from both sides.} \\
x^2-2x+4\left(y^2+6y\right) &= -33 && \text{Factor out leading coefficients.} \\
\left(x^2-2x+\underline{1}\right)+4\left(y^2+6y+\underline{9}\right) &= -33+\underline{1}+4(\underline{9}) && \text{Complete the Squares.} \\
(x-1)^2+4(y+3)^2 &= 4 && \text{Factor.} \\
\frac{(x-1)^2+4(y+3)^2}{4} &= \frac{4}{4} && \text{Divide through by 4} \\
\frac{(x-1)^2}{4}+(y+3)^2 &= 1 \\
\frac{(x-1)^2}{(2)^2}+\frac{(y-(-3))^2}{(1)^2} &= 1 && \text{Rewrite.}
\end{aligned}$$

We identify $h=1$ and $k=-3$ so our ellipse is centered at $(1,-3)$. With $a=2$, we move 2 units left and right from the center to get two points on the ellipse: $(-1,-3)$ and $(3,-3)$. Since $b=1$, we move 1 unit up and down from the center to obtain the points: $(1,-2)$ and $(1,-4)$.

Since we moved farther from the center in the x direction than in the y direction, the major axis will lie along the horizontal line $y=-3$ so the minor axis lies along the vertical line $x=1$. The vertices are the points on the ellipse which lie along the major axis so in this case, they are the points $(-1,-3)$ and $(3,-3)$, and the endpoints of the minor axis are $(1,-2)$ and $(1,-4)$.

To find the foci, we find $c=\sqrt{4-1}=\sqrt{3}$, which means the foci lie $\sqrt{3}$ units from the center. Since the major axis is horizontal, the foci lie $\sqrt{3}$ units to the left and right of the center, at $(1-\sqrt{3},-3)$ and $(1+\sqrt{3},-3)$. Plotting all of this information gives the graph below.



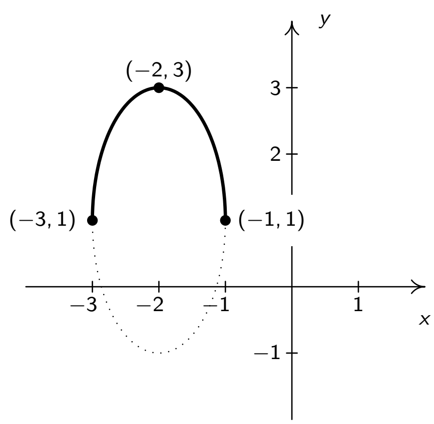
The graph of $x^2+4y^2-2x+24y+33=0$.

2. Recall to graph the function $f(x) = 1+2\sqrt{-x^2-4x-3}$, we graph the equation $y = 1+2\sqrt{-x^2-4x-3}$. We work to put this equation into standard form.

$$\begin{aligned}
y &= 1+2\sqrt{-x^2-4x-3} & \\
y-1 &= 2\sqrt{-x^2-4x-3} & \text{Isolate the radical term.} \\
(y-1)^2 &= \left(2\sqrt{-x^2-4x-3}\right)^2 & \text{Square both sides.} \\
(y-1)^2 &= 4(-x^2-4x-3) & \\
(y-1)^2 &= -4x^2-16x-12 & \\
4x^2+16x+(y-1)^2 &= -12 & \\
4\left(x^2+4x\right)+(y-1)^2 &= -12 & \text{Factor out leading coefficient of } x^2 \\
4\left(x^2+4x+\underline{4}\right)+(y-1)^2 &= -12+4(\,\underline{4}\,) & \text{Complete the Square in } x \\
4(x+2)^2+(y-1)^2 &= 4 & \text{Factor.} \\
\frac{4(x+2)^2+(y-1)^2}{4} &= \frac{4}{4} & \text{Divide through by 4.} \\
(x+2)^2+\frac{(y-1)^2}{4} &= 1 & \\
\frac{(x-(-2))^2}{(1)^2}+\frac{(y-1)^2}{(2)^2} &= 1 & \text{Rewrite.}
\end{aligned}$$

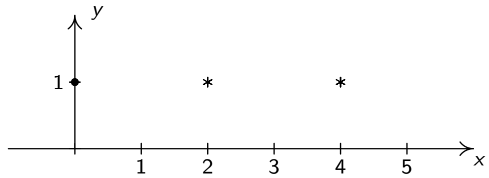
We identify $h=-2$ and $k=1$ so the ellipse is centered at $(-2,1)$. With $a=1$, we move 1 unit to the left and to the right an obtain the points $(-3,1)$ and $(-1,1)$. With $b=2$, we move 2 units up and down from the center to obtain two more points on the graph of the ellipse: $(-2,3)$ and $(-2,-1)$.

However, the graph of $f(x) = 1+2\sqrt{-x^2-4x-3}$ cannot be the *entire* ellipse, else it would violate the vertical line test. This means the graph of f must be a *portion* of the ellipse. Since, by definition $\sqrt{-x^2-4x-3} \geq 0$, we know $f(x) = 1+2\sqrt{-x^2-4x-3} \geq 1$. Hence, the graph of f must be the *upper* half of the ellipse, as denoted below on the right.



The graph of $f(x) = 1+2\sqrt{-x^2-4x-3}$.

3. (a) We plot the information given to us below and notice immediately that the major axis is horizontal, which means $a > b$. Since the center is the midpoint of the foci, we know the center of the ellipse is $(3, 1)$ which means $h = 3$ and $k = 1$. Since one vertex is at $(0, 1)$, which is 3 units from the center, we have that $a = 3$, so $a^2 = 9$. At this point, all that remains is to find b^2.



Since the foci, $(2, 1)$ and $(4, 1)$, are 1 unit away from the center, we have $c = 1$. Putting this together with the fact that $a > b$, we get $c = \sqrt{a^2 - b^2}$, or $1 = \sqrt{9 - b^2}$. Squaring both sides gives $1 = 9 - b^2$ or $b^2 = 8$. Hence, our final answer is:

$$\frac{(x-3)^2}{9} + \frac{(y-1)^2}{8} = 1.$$

(b) From the diagram, we infer the ellipse is taller than it is wide. More specifically, the labeled points $(0, 0)$ and $(6, 0)$ are the endpoints of the minor axis. This gives the center is $(3, 0)$, so $h = 3$ and $k = 0$. Moreover, we have $a = 3$.

While it certainly *appears* that the vertices are $(3, \pm 4)$, in which case we'd have $b = 4$, these points aren't labeled. Instead, we use the labeled point $(1, 3)$ to calculate b^2. At this stage, we know the equation of the ellipse is

$$\frac{(x-3)^2}{9} + \frac{y^2}{b^2} = 1,$$

so upon substituting $x = 1$ and $y = 3$, we obtain $\frac{4}{9} + \frac{9}{b^2} = 1$. Solving this equation, we get $b^2 = \frac{81}{5}$. Hence, our final answer is:

$$\frac{(x-3)^2}{9} + \frac{5y^2}{81} = 1.$$

Note the vertices of the ellipse are $\left(3, \pm\sqrt{\frac{81}{5}}\right) \approx (3, \pm 4)$ but they are not *exactly* $(3, \pm 4)$.